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LOGARITHMIC OPERATORS AND DYNAMICAL
EXTENSION OF THE SYMMETRY GROUP IN THE
BOSONIC $SU(2)_0$ AND SUSY $SU(2)_2$ WZNW MODELS.

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ABSTRACT

We study the operator product expansion in the bosonic $SU(2)_0$ and SUSY $SU(2)_2$ WZNW models. We find that these OPEs contain both logarithmic operators and new conserved currents, leading to an extension of the symmetry group.

1 Introduction

Dynamical generation of symmetries in quantum field theory is an intriguing subject. Such generation has recently been observed in the Wess-Zumino-Novikov-Witten (WZNW) model on the $U(r)$ group [1]. The latter model is a critical two-dimensional theory arising from the study of $2 + 1 - d$ relativistic fermions interacting with a random non-Abelian gauge potential, and possesses a Kac-Moody current algebra. It turned out that in the limit $r \rightarrow 0$ this theory acquires an additional conserved current acting as a generator of a continuous symmetry. This additional symmetry is related to the spectral degeneracy and emergence of the so-called “logarithmic operators” [2].

The logarithmic behaviour of multipoint correlation functions in two-dimensional conformal field theory was first discussed by Rozansky and Saleur [3]. They found such behaviour for the WZNW model on the supergroup $GL(1,1)$. Later the logarithms which once had been thought to be excluded altogether to preserve locality [4], have been found in a multitude of other models such as the gravitationally dressed CFTs [5], $c = -2$ and more general $c_{p,1}$ models [2] [6] [7] [8], the critical disordered models [1] [9], and might play a role in the study of critical polymers and percolation [10] [6] [7] [11] and $2D$ -magnetohydrodynamic turbulence [12]. They are also important for studying the problem of recoil in the theory of strings and D -branes [13] [14] [15] [16] [17] as well as target-space symmetries in string theory in general [13].

It was suggested by Gurarie [2] that logarithms appear when there is a degeneracy in the spectrum of conformal dimensions of the theory. A consequence is the appearance of additional operators which, together with ordinary primary ones, form the basis for the Jordan cell of the Virasoro operator L_0 . As we have already mentioned, in some cases spectral degeneracy has an additional effect producing extra symmetries.

In this paper we consider two types of critical models where such symmetries emerge. These are $SU(2)$ level $k = 0$ WZNW model and $SU(2)$ level $k = 2$ supersymmetric WZNW model. We approach these problems studying the operator product expansion (OPE) which follows from the solutions of Knizhnik-Zamolodchikov equations for multipoint correlation functions of the primary fields. We establish that at $k = 0$ ($k = 2$ for the SUSY theory) the emerging OPE contains additional current operators K^a and N^a ($a = 1, 2, 3$). Therefore at these special values of k the current algebra enlarges to include 6 additional

currents. We use these particular models as the simplest examples of two general classes - the $SU(N)$ level $k = 0$ WZNW model and $SU(N)$ level $k = N$ supersymmetric WZNW model. It can be shown that the only difference is that there are $2(N^2 - 1)$ new currents in the $SU(N)$ case instead of the 6 currents we shall consider in this paper. The SUSY case $SU(2)$ at level $k = 2$ was briefly discussed earlier in the context of solitons in string theory [13]

2 Formulation of the theory

The generators for the Kac-Moody and Virasoro algebras for a 2-d conformal field theory with zero central extension have the OPEs

$$\begin{aligned} T(z_1)T(z_2) &= \frac{c/2}{z_{12}^4} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}}\partial T(z_2) + \dots \\ T(z_1)J^a(z_2) &= \frac{1}{z_{12}^2}J^a(z_1) + \frac{1}{z_{12}}\partial J^a(z_2) + \dots \\ J^a(z_1)J^b(z_2) &= \frac{f^{abc}}{z_{12}}J^c(z_2) + \dots \end{aligned} \quad (1)$$

We define our WZNW model via the Sugawara construction

$$T(z) = \frac{1}{2\kappa} : J^a(z)J^a(z) : \quad (2)$$

and take the gauge group to be $SU(2)$. Elementary calculations show that $\kappa = -\frac{c_v}{2}$ where $c_v\delta^{cd} = f^{abc}f^{abd}$, and that the central charge of the Virasoro algebra vanishes.

We assume the existence of primary fields transforming as

$$\begin{aligned} T(z_1)\phi(z_2, \bar{z}_2) &= \frac{\Delta}{z_{12}^2}\phi(z_2, \bar{z}_2) + \frac{1}{z_{12}}\partial\phi(z_2, \bar{z}_2) + \dots \\ J^a(z_1)\phi(z_2, \bar{z}_2) &= \frac{t^a}{z_{12}}\phi(z_2, \bar{z}_2) + \dots \end{aligned} \quad (3)$$

In terms of the mode expansions $T(z) = \sum_n L_n/z^{n+2}$ and $J^a(z) = \sum_n J_n^a/z^{n+1}$ the relation (2) reads

$$2L_n = \sum_{m=-\infty}^{\infty} : J_m^a J_{n-m}^a :$$

with conventional normal ordering.

Applying this to g for $n = -1$, we find the existence of the null vector

$$(J_{-1}^a t^a - L_{-1})\phi = 0 \quad (4)$$

and see that the primary field has dimension $\Delta = 3/8$. These, and the Ward Identities of conformal field theory, directly lead to the Knizhnik-Zamolodchikov equations [18] for our case:

$$\left\{ \frac{c_v}{2} \partial_{z_i} + \sum_{i \neq j=1}^N \frac{t_i^a t_j^a}{z_{ij}} \right\} < |\phi(z_1, \bar{z}_1) \dots \phi(z_N, \bar{z}_N)| > = 0 \quad (5)$$

The full correlator can be built up starting from the chiral conformal blocks by solving the monodromy problem to ensure locality [19]. This, contrary to what was anticipated a long time ago, can in some special cases be done in a consistent way.

The conformal blocks for the level zero KZ equations were studied in the context of a more general chain of thought in [20]. There, an expression was obtained for the general $2n$ -point conformal blocks. The fields involved are taken to be from the fundamental spin-1/2 representation, and thus carry an index ϵ . For a general conformal block of order $2n$, the multiindex $\epsilon_1, \dots, \epsilon_{2n}$ has to obey $\sum \epsilon_i = 0$ since we are looking for singlet solutions of the equations. There exists then a partition of $B = \{1, \dots, 2n\}$ into $T = \{i_k\}_{k=1}^n : \epsilon_{i_k} = +$ and $T' = \{j_k\}_{k=1}^n : \epsilon_{j_k} = -$. Distinct solutions are parametrized by the sets $\gamma_1, \dots, \gamma_{n-1}$.

$$f_{\gamma_1, \dots, \gamma_{n-1}}^{\epsilon_1, \dots, \epsilon_{2n}}(\lambda_1, \dots, \lambda_{2n}) = \prod_{i < j} (\lambda_i - \lambda_j)^{1/4} \times \prod_{k \in T, l \in T'} (\lambda_k - \lambda_l)^{-1} \det \left| \int_{\gamma_p} \zeta_q(\tau | T | T') \right|_{(n-1) \times (n-1)} \quad (6)$$

where ζ_q are given by the following differentials on the hyper-elliptic surface (HES) $w^2 = P(\tau) = \prod(\tau - \lambda_i)$:

$$\begin{aligned} \zeta_q(\tau | T | T') &= \frac{Q_q(\tau | T | T')}{\sqrt{P(\tau)}} \\ Q_q(\tau) &= \prod_{i \in T} (\tau - \lambda_i) \left[\frac{d}{d\tau} \frac{\prod_{j \in T'} (\tau - \lambda_j)}{\tau^{n-q}} \right]_0 + \\ &+ \prod_{i \in T'} (\tau - \lambda_i) \left[\frac{d}{d\tau} \frac{\prod_{j \in T} (\tau - \lambda_j)}{\tau^{n-q}} \right]_0 \end{aligned} \quad (7)$$

and where the $\gamma_i, i = 1, \dots, n-1$ contours are chosen to be either of the set $\{a_i, b_i\}$: a_i surrounds the cut running from λ_{2i-1} and λ_{2i} , and b_i starts from one bank of the cut $\lambda_{2i-1}, \lambda_{2i}$, reaches the cut $\lambda_{2g+1}, \lambda_{2g+2}$ and returns to the other bank of $\lambda_{2i-1}, \lambda_{2i}$ by another sheet (the prescription is given in [21]).

Given these, we can in principle study the full algebra of the theory. We start

this endeavour in the next section, and come to some surprises quite soon along the way.

3 Four-point correlation functions and Operator Product Expansions

The integral representation (6) allows us to calculate the conformal blocks

$$f^{\epsilon_1 \dots \epsilon_4}(z_1, z_2, z_3, z_4) \sim \langle |V_{\epsilon_1}(z_1) \dots V_{\epsilon_4}(z_4)| \rangle \quad (8)$$

where V is our primary chiral field, and $|\rangle$ is the $sl(2, C)$ -invariant vacuum from which we will build our physical Kac-Moody invariant correlator.

As usual, since we want our correlation function to be invariant under the Kac-Moody algebra, we decompose our correlator in terms of amplitudes multiplied on the independent invariant tensors (basic singlets) of the representation. The two independent contours $\gamma = a = [0, z]$ and $\gamma = b = [z, 1]$ then generate the independent solutions:

$$\begin{aligned} f_{\gamma}^{\epsilon_1 \dots \epsilon_4}(z_1, z_2, z_3, z_4) &= \frac{1}{[z_{13}z_{24}]^{3/4}} f_{\gamma}^{\epsilon_1 \dots \epsilon_4}(z) \\ f_{\gamma}^{\epsilon_1 \dots \epsilon_4}(z) &= [z(1-z)]^{1/4} \sum_{A=1}^2 J_A F_{\gamma}^A(z) \\ J_1 &= \delta_{\epsilon_1 \epsilon_2} \delta_{\epsilon_3 \epsilon_4}, \quad J_2 = \delta_{\epsilon_1 \epsilon_4} \delta_{\epsilon_2 \epsilon_3} \end{aligned} \quad (9)$$

where

$$\begin{aligned} F_a^1(z) &= -\frac{1}{2} F(1/2, 3/2; 1; z) \\ F_b^1(z) &= \frac{\pi}{4} F(1/2, 3/2; 2; 1-z) \\ &= -\frac{1}{2} \ln z F(1/2, 3/2; 1; z) - \frac{1}{2} H_0(z) \\ F_a^2(z) &= -\frac{1}{4} F(1/2, 3/2; 2; z) \\ F_b^2(z) &= \frac{\pi}{2} F(1/2, 3/2; 1; 1-z) \\ &= \frac{1}{z} - \frac{1}{4} \ln z F(1/2, 3/2; 2; z) - \frac{1}{4} H_1(z) \\ H_i(z) &= \sum_{n=0}^{\infty} z^n \frac{(1/2)_n (3/2)_n}{n! (n+i)!} \times \{ \Psi(1/2+n) + \\ &\quad + \Psi(3/2+n) - \Psi(n+1) - \Psi(n+i+1) \} \end{aligned} \quad (10)$$

The functions F_b^i have logarithmic behavior near $z = 0$:

$$\begin{aligned}
F_a^1(z) &= -\frac{1}{2} \left(1 + \frac{3}{4}z + O(z^2) \right) \\
F_b^1(z) &= -\frac{1}{2} \ln z + \frac{\alpha}{2} + O(z \ln z) \\
F_a^2(z) &= -\frac{1}{4} \left(1 + \frac{3}{8}z + O(z^2) \right) \\
F_b^2(z) &= \frac{1}{z} - \frac{1}{4} \ln z + \frac{\alpha + 1}{4} + O(z \ln z)
\end{aligned} \tag{11}$$

where $\alpha = 4 \ln 2 - 2$.

We wish for a little while to concentrate on the logarithmic solutions. To accommodate the logarithms in these we have to postulate for the primary fields an OPE containing logarithmic operators:

$$V_{\epsilon_1}(z_1)V_{\epsilon_2}(z_2) = \frac{1}{z_{12}^{3/4}} \left\{ I_{\epsilon_2 \epsilon_2^\vee} - z_{12} t_{\epsilon_2 \epsilon_2^\vee}^i \left[D^i(z_2) + \ln z_{12} C^i(z_2) \right] + \dots \right\} \tag{12}$$

where I is the unit matrix and ϵ^\vee is the weight conjugate to ϵ . Taking the four-point vertex function and performing the fusion, we obtain

$$\begin{aligned}
\langle |V_{\epsilon_1}(z_1)V_{\epsilon_2}(z_2)D^i(z_3)| \rangle &= t_{\epsilon_1 \epsilon_2^\vee}^i \left[\alpha - \ln \frac{z_{12}}{z_{13}z_{23}} \right] \frac{z_{12}^{1/4}}{z_{13}z_{23}} \\
\langle |V_{\epsilon_1}(z_1)V_{\epsilon_2}(z_2)C^i(z_3)| \rangle &= -t_{\epsilon_1 \epsilon_2^\vee}^i \frac{z_{12}^{1/4}}{z_{13}z_{23}}
\end{aligned} \tag{13}$$

Fusing further, we get the two-point functions

$$\begin{aligned}
\langle |D^i(z_1)D^j(z_2)| \rangle &= -[\alpha + 2 \ln z_{24}] \frac{\delta^{ij}}{z_{12}^2} \\
\langle |C^i(z_1)D^j(z_2)| \rangle &= \frac{\delta^{ij}}{z_{12}^2} \\
\langle |C^i(z_1)C^j(z_2)| \rangle &= 0
\end{aligned} \tag{14}$$

We can now look at the action of the Virasoro algebra generators on these structures. We start from the commutation relation between the Virasoro generators and a primary field:

$$[L_k, B(z)] = \left(z^{k+1} \partial + (k+1) \Delta_B z^k \right) B(z) \tag{15}$$

Then, by mode-expanding our operators

$$C^i(z) = \sum \frac{C_n^i}{z^{n+1}} \quad D^i(z) = \sum \frac{D_n^i}{z^{n+1}} \tag{16}$$

we get the commutation relations

$$\begin{aligned}
[L_0, C_{-n}^i] &= (n+1)C_{-n}^i \\
[L_0, D_{-n}^i] &= (n+1)D_{-n}^i + C_{-n}^i \\
[L_k, C_{-n-k}^i] &= \left(n+1 + \frac{3}{8}(k-1)\right) C_{-n}^i \\
[L_k, D_{-n-k}^i] &= \left(n+1 + \frac{3}{8}(k-1)\right) D_{-n}^i + C_{-n}^i
\end{aligned} \tag{17}$$

for $k \geq 1$. The operator L_0 thus has the operators C and D as a basis for its Jordan cell [2]. These allow to express the states $|C^i, n\rangle$ and $|D^i, n\rangle$ in terms of the descendants of $|C^i, 0\rangle$ and $|D^i, 0\rangle$. The result for the first two levels is

$$\begin{aligned}
|C^i, 1\rangle &= \frac{1}{2}L_{-1}|C^i, 0\rangle \\
|C^i, 2\rangle &= \left[\frac{3}{8}L_{-2} - \frac{1}{48}L_{-1}^2\right]|C^i, 0\rangle \\
|D^i, 1\rangle &= \frac{1}{2}L_{-1}|D^i, 0\rangle \\
|D^i, 2\rangle &= \left[\frac{3}{8}L_{-2} - \frac{1}{48}L_{-1}^2\right]|D^i, 0\rangle + \left[-\frac{23}{24}L_{-2} + \frac{83}{144}L_{-1}^2\right]|C^i, 0\rangle
\end{aligned} \tag{18}$$

We are interested in the “full” four-point correlation functions, in which the anti-holomorphic dependence has been reestablished:

$$G(z_i, \bar{z}_i) = \langle |g(z_1, \bar{z}_1)g^\dagger(z_2, \bar{z}_2)g(z_3, \bar{z}_3)g^\dagger(z_4, \bar{z}_4)| \rangle$$

for the $SU(2)_0$ WZNW model. We write our primary field as $g_{\epsilon\bar{\epsilon}}(z, \bar{z})$ where the indices $\epsilon(\bar{\epsilon})$ correspond respectively to the left (right) $SU(2)$ group: i.e., in the above, $g(z, \bar{z}) = g_{\epsilon\bar{\epsilon}}(z, \bar{z})$; $g^\dagger(z, \bar{z}) = g_{\bar{\epsilon}\epsilon}^\dagger(z, \bar{z})$. Because of the conformal Ward identities [18], we can give the following form to the four-point function:

$$\begin{aligned}
&\langle |g_{\epsilon_1\bar{\epsilon}_1}(z_1, \bar{z}_1)g_{\bar{\epsilon}_2\epsilon_2}^\dagger(z_2, \bar{z}_2)g_{\epsilon_3\bar{\epsilon}_3}(z_3, \bar{z}_3)g_{\bar{\epsilon}_4\epsilon_4}^\dagger(z_4, \bar{z}_4)| \rangle = \\
&\quad |(z_1 - z_3)(z_2 - z_4)|^{-4\Delta} G_{\epsilon_1\bar{\epsilon}_1, \dots, \epsilon_4\bar{\epsilon}_4}(z, \bar{z}) \\
G_{\epsilon_1\bar{\epsilon}_1, \dots, \epsilon_4\bar{\epsilon}_4}(z, \bar{z}) &:= \lim_{|z_\infty| \rightarrow \infty} |z_\infty|^{4\Delta} \langle |g_{\epsilon_1\bar{\epsilon}_1}(0, 0)g_{\bar{\epsilon}_2\epsilon_2}^\dagger(z, \bar{z})g_{\epsilon_3\bar{\epsilon}_3}(1, 1)g_{\bar{\epsilon}_4\epsilon_4}^\dagger(z_\infty, \bar{z}_\infty)| \rangle
\end{aligned} \tag{19}$$

where $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$. We can here again develop our correlator in the same fashion as usual:

$$\begin{aligned}
G_{\epsilon_1\bar{\epsilon}_1 \dots \epsilon_4\bar{\epsilon}_4}(z, \bar{z}) &= |z(1-z)|^{1/2} \sum_{i,j=1,2} (I_i)_{\epsilon_1, \dots, \epsilon_4} (\bar{I}_j)_{\bar{\epsilon}_1, \dots, \bar{\epsilon}_4} G^{ij}(z, \bar{z}) \\
I_1 &= \delta_{\epsilon_1\epsilon_4} \delta_{\epsilon_2\epsilon_3} \quad I_2 = \delta_{\epsilon_1\epsilon_2} \delta_{\epsilon_3\epsilon_4}
\end{aligned} \tag{20}$$

To build the physical correlator, we have to solve the monodromy problem to ensure locality. The conformal blocks have to obey the crossing symmetry

$$G_{\epsilon_1 \bar{\epsilon}_1 \epsilon_2 \bar{\epsilon}_2 \epsilon_3 \bar{\epsilon}_3 \epsilon_4 \bar{\epsilon}_4}(z, \bar{z}) = G_{\epsilon_1 \bar{\epsilon}_1 \epsilon_4 \bar{\epsilon}_4 \epsilon_3 \bar{\epsilon}_3 \epsilon_2 \bar{\epsilon}_2}(1-z, 1-\bar{z}) \quad (21)$$

which leads to the construction

$$\begin{aligned} G^{ij}(z, \bar{z}) &= \sum_{p,q=a,b} U^{pq} F_p^i(z) F_q^j(\bar{z}) \\ U^{ab} &= U^{ba} = 1 \quad U^{aa} = U^{bb} = 0 \end{aligned} \quad (22)$$

Single-valuedness around $z = 0$ and $z = 1$ has respectively necessitated $U^{bb} = 0$ and $U^{aa} = 0$, because we cannot allow terms of the form $\ln z \ln \bar{z}$. It is interesting to note that this will lead directly to the absence of the unit operator in the OPE of two primary fields g , and thus to the vanishing of their two-point correlation function. We have to switch the sets of indices because our correlation function involves g^\dagger at z_2 and z_4 . We thus have the following behaviors near $z = 0$:

$$\begin{aligned} G^{11}(z, \bar{z}) &= \frac{1}{2} \ln |z| - \frac{\alpha}{2} + O(z \ln z) \\ G^{12}(z, \bar{z}) &= -\frac{1}{2\bar{z}} + \frac{1}{4} \ln |z| - \frac{\alpha}{4} - \frac{1}{8} + O(z \ln z) \\ G^{22}(z, \bar{z}) &= -\frac{1}{4z} - \frac{1}{4\bar{z}} + \frac{1}{8} \ln |z| - \frac{\alpha+1}{8} + O(z \ln z) \end{aligned} \quad (23)$$

The dominant terms can be accounted for with the OPE

$$\begin{aligned} g_{\epsilon_1 \bar{\epsilon}_1}(z_1, \bar{z}_1) g_{\bar{\epsilon}_2 \epsilon_2}^\dagger(z_2, \bar{z}_2) &= |z_{12}|^{-3/2} \times \\ &\times \left\{ z_{12} \delta_{\bar{\epsilon}_1 \bar{\epsilon}_2} t_{\epsilon_1 \epsilon_2}^i K^i(z_2) + O(z_{12}^2) \right. \\ &\quad \left. + \bar{z}_{12} \delta_{\epsilon_1 \epsilon_2} \bar{t}_{\bar{\epsilon}_1 \bar{\epsilon}_2}^i \bar{K}^i(\bar{z}_2) + O(\bar{z}_{12}^2) \right. \\ &\quad \left. + |z_{12}|^2 t_{\epsilon_1 \epsilon_2}^i \bar{t}_{\bar{\epsilon}_1 \bar{\epsilon}_2}^j \left[D^{ij}(z_2, \bar{z}_2) + \ln |z_{12}| C^{ij}(z_2, \bar{z}_2) \right] + \dots \right\} \end{aligned} \quad (24)$$

In this, we use the set of matrices $\{t^a\}$, $a = 1, 2, 3$ defined by

$$t^a = \frac{\sigma^a}{2}, \quad [t^a, t^b] = i\epsilon^{abc} t^c \quad (25)$$

The term of $O(1/|z|^2)$ is absent in G^{11} because we were forced to take $U^{bb} = 0$. As anticipated above, this prevents the appearance of the unit operator in (25). Consistency thus leads to

$$\langle |g_{\epsilon_1 \bar{\epsilon}_1}(z_1, \bar{z}_1) g_{\bar{\epsilon}_2 \epsilon_2}^\dagger(z_2, \bar{z}_2)| \rangle = 0 \quad (26)$$

The appearance of the logarithm, as Gurarie has taught us, is due to the presence of operators in the OPE (25) whose dimensions become degenerate. There is an enlightening way to see how this comes about, namely by taking the expression for the full correlator for finite k [18] and then going to the limit $k \rightarrow 0$. The solution reads

$$\begin{aligned}
G(z_i, \bar{z}_i) &= \frac{1}{|z_{13}z_{24}|^{4\Delta}} G(z, \bar{z}) & \Delta &= \frac{3}{4(2+k)} \\
G_{AB}(z, \bar{z}) &= \sum_{p,q=0,1} U_{pq} F_A^{(p)}(z) F_B^{(q)}(\bar{z}) \\
F_1^{(0)}(z) &= x^{-\frac{3}{2(2+k)}} (1-x)^{\frac{1}{2(2+k)}} F\left(\frac{1}{2+k}, -\frac{1}{2+k}; \frac{k}{2+k}; z\right) \\
F_2^{(0)}(z) &= \frac{1}{k} x^{\frac{1+2k}{2(2+k)}} (1-x)^{\frac{1}{2(2+k)}} F\left(\frac{1+k}{2+k}, \frac{3+k}{2+k}; \frac{2+2k}{2+k}; z\right) \\
F_1^{(1)}(z) &= x^{\frac{1}{2(2+k)}} (1-x)^{\frac{1}{2(2+k)}} F\left(\frac{1}{2+k}, \frac{3}{2+k}; \frac{4+k}{2+k}; z\right) \\
F_2^{(1)}(z) &= -2x^{\frac{1}{2(2+k)}} (1-x)^{\frac{1}{2(2+k)}} F\left(\frac{1}{2+k}, \frac{3}{2+k}; \frac{2}{2+k}; z\right) \\
U_{10} &= U_{01} = 0 & U_{11} &= h U_{00} \\
h &= \frac{1}{4} \frac{\Gamma(\frac{1}{2+k}) \Gamma(\frac{3}{2+k})}{\Gamma(\frac{1+k}{2+k}) \Gamma(\frac{-1+k}{2+k})} \frac{\Gamma^2(\frac{k}{2+k})}{\Gamma^2(\frac{2}{2+k})} \quad (27)
\end{aligned}$$

When we take the limit $k \rightarrow 0$, we have to deal with gamma functions of vanishing arguments, and hypergeometric functions of vanishing γ (third parameter). The former is easily tackled using the recursion relation

$$\Gamma(x+1) = x\Gamma(x)$$

giving, for example,

$$\Gamma\left(\frac{k}{2+k}\right) = \frac{2}{k} + 1 + \psi(1) + O(k)$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$, whereas the latter necessitates the use of standard identities like

$$\gamma(\gamma+1)F(\alpha, \beta; \gamma; z) = \gamma(\gamma+1)F(\alpha, \beta; \gamma+1; z) + \alpha\beta z F(\alpha+1, \beta+1; \gamma+2; z) \quad (28)$$

to yield, for example,

$$\begin{aligned}
F\left(\frac{1}{2+k}, -\frac{1}{2+k}; \frac{k}{2+k}; z\right) &= -\frac{1}{2k} z F(1/2, 3/2; 2; z) + F(1/2, 1/2; 1; z) - \\
&\quad - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n (3/2)_n}{(2)_n} \frac{z^n}{n!} \left[-\frac{1}{4n+2} + \frac{1}{2} - \frac{1}{2} [\psi(n+2) - \psi(2)] \right] + O(k) \quad (29)
\end{aligned}$$

When writing down the full correlator as a power series in k , the dominant term is of order $1/k^2$, but adds up to order $1/k$ with the identity

$$\lim_{k \rightarrow 0} \frac{z^k - 1}{k} = \ln z$$

Doing the expansion teaches us that there are operators in the OPE of g, g^\dagger of dimensions $\Delta = \bar{\Delta} = \frac{2}{2+k}$ and 1, i.e.

$$g(1)g^\dagger(2) = \dots + |z_{12}|^{\frac{1}{2+k}} t^i t^j A^{ij}(2) + |z_{12}|^{\frac{1+2k}{2+k}} t^i t^j B^{ij}(2) + \dots$$

which become degenerate as $k \rightarrow 0$, yielding the logarithmic pair. Thus, the divergent term (of order $1/k$) in the expansion is just our previous solution, as can be straightforwardly checked, and the unit operator in (25) is suppressed as $k \rightarrow 0$ if we take a finite four-point function.

From all this, we can read the two-point functions between the newly introduced fields:

$$\begin{aligned} \langle |K^i(z_1)K^j(z_2)| \rangle &= \frac{\delta^{ij}}{z_{12}^2} \\ \langle |\bar{K}^i(\bar{z}_1)\bar{K}^j(\bar{z}_2)| \rangle &= \frac{\delta^{ij}}{\bar{z}_{12}^2} \\ \langle |C^{ik}(z_1, \bar{z}_1)D^{jl}(z_2, \bar{z}_2)| \rangle &= \frac{2\delta^{ij}\delta^{kl}}{|z_{12}|^4} \\ \langle |D^{ik}(z_1, \bar{z}_1)D^{jl}(z_2, \bar{z}_2)| \rangle &= -2[\alpha + 2\ln|z_{12}|] \frac{\delta^{ij}\delta^{kl}}{|z_{12}|^4} \end{aligned} \quad (30)$$

with all other correlators vanishing.

We want to look now at the OPE's involving the Kac-Moody current. The Kac-Moody current with zero central extension obeys the OPE

$$J^a(z_1)J^b(z_2) = \frac{i\epsilon^{abc}}{z_{12}} J^c(z_2) \quad (31)$$

The Ward Identity involving the Kac-Moody current then reads

$$\begin{aligned} &\langle |J^a(z)J^{b_1}(\zeta_1)\dots J^{b_n}(\zeta_n)g(1)g^\dagger(2)g(3)g^\dagger(4)| \rangle = \\ &= \sum_{k=1}^n \frac{i\epsilon^{ab_k c}}{z - \zeta_k} \langle |J^{b_1}(\zeta_1)\dots J^{b_{k-1}}(\zeta_{k-1})J^c(\zeta_k)\dots J^{b_n}(\zeta_n)g(1)g^\dagger(2)g(3)g^\dagger(4)| \rangle + \\ &\quad + \sum_{j=1}^4 \frac{t_{(j)}^a}{z - z_j} \langle |J\dots Jg(1)g^\dagger(2)g(3)g^\dagger(4)| \rangle \end{aligned} \quad (32)$$

where the matrices $t_{(j)}^a$ act on the left indices of the fields g (we have to remember that there is a sign difference between the variations of g and g^\dagger under K-M transformations, which shows up implicitly in the Ward Identity).

A straightforward calculation gives

$$\begin{aligned}
\langle |J^a(z_1)K^b(z_2)K^c(z_3)| \rangle &= \frac{i\epsilon^{abc}}{z_{12}z_{13}z_{23}} \\
\langle |J^a(z_1)J^b(z_2)K^c(z_3)K^d(z_4)| \rangle &= \frac{\delta^{ab}\delta^{cd}z_{12}z_{34} - \delta^{ac}\delta^{bd}z_{13}z_{24} + \delta^{ad}\delta^{bc}z_{14}z_{23}}{z_{12}z_{13}z_{14}z_{23}z_{24}z_{34}} \\
\langle |J^i(z_1)J^j(z_2)N^k(z_3)| \rangle &= \frac{i\epsilon^{ijk}}{z_{12}z_{13}z_{23}} \tag{33}
\end{aligned}$$

where we have introduced still the other field N for consistency. From this we can get the following set of OPEs:

$$\begin{aligned}
K^i(z_1)K^j(z_2) &= \frac{\delta^{ij}}{z_{12}^2} + \frac{i\epsilon^{ijk}}{z_{12}}N^k(z_2) + \dots \\
J^i(z_1)J^j(z_2) &= \frac{i\epsilon^{ijk}}{z_{12}}J^k(z_2) + \dots \\
J^i(z_1)K^j(z_2) &= \frac{i\epsilon^{ijk}}{z_{12}}K^k(z_2) + \dots \\
J^i(z_1)N^j(z_2) &= \frac{\delta^{ij}}{z_{12}^2} + \frac{i\epsilon^{ijk}}{z_{12}}N^k(z_2) + \dots \tag{34}
\end{aligned}$$

The other OPEs are still undetermined at this point: we would need still the OPEs for KN and NN , which entail knowledge of respectively the six-point and eight-point correlation functions of gs . We have not worked out these functions at this stage.

4 Supersymmetric WZW model

The WZW model for $SU(2)$ at level k is equivalent to the bosonic sector of the supersymmetric WZW (SWZW) model at level $k+2$ [22, 23], and so we can use the results of the previous section to find the conformal blocks and OPE of the supersymmetric model at level 2. Unlike the bosonic $k=0$ model, the SWZW model at $k=2$ has a non-zero action, which can be written as (we use the notation of [23]):

$$S = \frac{k}{16\pi} \int dz d\bar{z} d\theta d\bar{\theta} \left[-Dg^s \bar{D}g^{s\dagger} + \int dt g^s \frac{\partial g^{s\dagger}}{\partial t} (Dg^s \bar{D}g^{s\dagger} + \bar{D}g^s Dg^{s\dagger}) \right] \tag{35}$$

Here g^s is a superfield in the fundamental representation of $SU(2)$. The super-Virasoro algebra is generated by the super-stress tensor $T^s(z, \theta)$, which has the OPE

$$T^s(Z_1)T^s(Z_2) = \frac{\hat{c}}{4Z_{12}^3} + \frac{3\theta_{12}}{2Z_{12}^2}T^s(Z_2) + \frac{1}{2Z_{12}}DT^s(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial T^s(Z_2) + \dots \quad (36)$$

For the model (35) with $k = 2$, the superconformal charge $\hat{c} = 1$. There is also a super-Kac-Moody algebra, generated by supercurrents J^{sa} , with the the OPEs:

$$\begin{aligned} T^s(Z_1)J^{sa}(Z_2) &= \frac{\theta_{12}}{2Z_{12}^2}J^{sa}(Z_2) + \frac{1}{2Z_{12}}DJ^{sa}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial J^{sa}(Z_2) + \dots \\ J^{sa}(Z_1)J^{sb}(Z_2) &= \frac{\delta^{ab}}{Z_{12}} + \frac{\theta_{12}}{Z_{12}}f^{abc}J^{sc}(Z_2) + \dots \end{aligned} \quad (37)$$

In these formulas, $Z = (z, \theta)$, $\theta_{12} = \theta_1 - \theta_2$ and $Z_{12} = z_1 - z_2 - \theta_1\theta_2$. $D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$ and $\partial = \frac{\partial}{\partial z}$. T^s and J^{sa} are superfields which contain the bosonic stress tensor and currents:

$$\begin{aligned} T^s(Z) &= \Theta(z) + \theta T(z) \\ J^{sa}(Z) &= j^a(z) + \theta J^a(z) \end{aligned} \quad (38)$$

We are interested in the four-point correlation function of g^s . It was shown in [23] that, as a consequence of the superconformal Ward Identities, the four-point function in the SWZW model at level k can be expressed in terms of the corresponding function in the WZW model at level $k - 2$. In the case of $k = 2$, the result is:

$$\langle |g^s(Z_1, \bar{Z}_1)g^{s\dagger}(Z_2, \bar{Z}_2)g^s(Z_3, \bar{Z}_3)g^{s\dagger}(Z_4, \bar{Z}_4)| \rangle = |Z_{13}Z_{24}|^{-4\Delta}\mathcal{G}(Z, \theta, \theta', \bar{Z}, \bar{\theta}, \bar{\theta}') \quad (39)$$

$$\begin{aligned} \mathcal{G}(Z, \theta, \theta', \bar{Z}, \bar{\theta}, \bar{\theta}') &= |Z(1-Z)|^{1/2} \sum_{i,j=1,2} I_i \bar{I}_j \mathcal{G}^{ij}(Z, \theta, \theta', \bar{Z}, \bar{\theta}, \bar{\theta}') \\ I_1 &= \delta_{\epsilon_1\epsilon_4}\delta_{\epsilon_2\epsilon_3} \quad I_2 = \delta_{\epsilon_1\epsilon_2}\delta_{\epsilon_3\epsilon_4} \end{aligned} \quad (40)$$

$$\mathcal{G}^{ij}(Z, \theta, \theta', \bar{Z}, \bar{\theta}, \bar{\theta}') = [1 + \mathcal{Q}(1-Z)^{-1/2}\theta\theta']G^{ij}(Z, \bar{Z})[1 + \mathcal{Q}^T(1-Z)^{-1/2}\theta\theta']$$

Where

$$\begin{aligned} Z &= \frac{Z_{12}Z_{34}}{Z_{13}Z_{24}} \\ \theta &= \theta_{124} \quad \theta' = \theta_{123} \\ \theta_{ijk} &= (Z_{ij}Z_{jk}Z_{ki})^{-1/2}(\theta_i Z_{jk} + \theta_j Z_{ki} + \theta_k Z_{ij} + \theta_i\theta_j\theta_k) \\ \mathcal{Q} &= \begin{pmatrix} 3/4 & 0 \\ 1/2 & -1 \end{pmatrix} \end{aligned} \quad (41)$$

$G^{ij}(Z, \bar{Z})$ is the same function as in (20). The θ -independent part of the four-point function in the supersymmetric model is therefore identical to the four-point function in the bosonic model. The OPE for the supersymmetric model is therefore essentially the same as (25):

$$\begin{aligned}
g_{\epsilon_1 \bar{\epsilon}_1}^s(Z_1, \bar{Z}_1) g_{\epsilon_2 \bar{\epsilon}_2}^{s\dagger}(Z_2, \bar{Z}_2) &= |Z_{12}|^{-3/2} \times \\
&\times \left\{ Z_{12} \delta_{\bar{\epsilon}_1 \bar{\epsilon}_2} t_{\epsilon_1 \epsilon_2}^i K^{si}(Z_2) + \bar{Z}_{12} \delta_{\epsilon_1 \epsilon_2} \bar{t}_{\bar{\epsilon}_1 \bar{\epsilon}_2}^i \bar{K}^{si}(\bar{Z}_2) + \right. \\
&+ |Z_{12}|^2 t_{\epsilon_1 \epsilon_2}^i \bar{t}_{\bar{\epsilon}_1 \bar{\epsilon}_2}^j \left[D^{sij}(Z_2, \bar{Z}_2) + \ln |Z_{12}| C^{sij}(Z_2, \bar{Z}_2) \right] + \dots \left. \right\}
\end{aligned} \tag{42}$$

The non-zero two-point functions are:

$$\begin{aligned}
\langle |K^{si}(Z_1) K^{sj}(Z_2)| \rangle &= \frac{\delta^{ij}}{Z_{12}^2} \\
\langle |\bar{K}^{si}(\bar{Z}_1) \bar{K}^{sj}(\bar{Z}_2)| \rangle &= \frac{\delta^{ij}}{\bar{Z}_{12}^2} \\
\langle |C^{sik}(Z_1, \bar{Z}_1) D^{sjl}(Z_2, \bar{Z}_2)| \rangle &= \frac{2\delta^{ij}\delta^{kl}}{|Z_{12}|^4} \\
\langle |D^{sik}(Z_1, \bar{Z}_1) D^{sjl}(Z_2, \bar{Z}_2)| \rangle &= -2[\alpha + 2\ln |Z_{12}|] \frac{\delta^{ij}\delta^{kl}}{|Z_{12}|^4}
\end{aligned} \tag{43}$$

The Ward Identity involving J^{sa} is similar to (32)

$$\langle |J^{sa}(Z_0) g^s(1) g^{s\dagger}(2) \dots g^{s\dagger}(2n)| \rangle = \sum_{j=1}^{2n} \frac{\theta_{0j} t_{(j)}^a}{Z_{0j}} \langle |g^s(1) g^{s\dagger}(2) \dots g^{s\dagger}(2n)| \rangle \tag{44}$$

From (44) with $n = 2$ and (43), we can calculate the three-point function:

$$\langle |J^{sa}(Z_1) K^{sb}(Z_2) K^{sc}(Z_3)| \rangle = \frac{-i\epsilon^{abc}\theta_{123}}{Z_{12}^{1/2} Z_{31}^{1/2} Z_{23}^{3/2}} \tag{45}$$

Of course (43) and (45) are just the supersymmetric generalisations of (30) and (33). To calculate correlation functions involving two factors of the super-current J^{sa} , we have to take into account the central extension in the OPE

$$J^{sa}(Z_1) J^{sb}(Z_2) = \frac{\delta^{ab}}{Z_{12}} + \frac{\theta_{12}}{Z_{12}} i\epsilon^{abc} J^{sc}(Z_2) + \dots \tag{46}$$

This leads to the correlation function:

$$\begin{aligned}
\langle |J^{sa}(Z_1) J^{sb}(Z_2) K^{sc}(Z_3) K^{sd}(Z_4)| \rangle &= \frac{\delta^{ab}\delta^{cd}}{Z_{12}Z_{34}^2} + \\
+ Z_{12}^{-1/2} Z_{13}^{-1/2} Z_{14}^{-1/2} Z_{23}^{-1/2} Z_{24}^{-1/2} Z_{34}^{-3/2} &\times \left[\delta^{ab}\delta^{cd} Z_{12}^{1/2} Z_{34}^{1/2} \theta_{134} \theta_{234} \right. \\
&+ \delta^{ac}\delta^{bd} Z_{13}^{1/2} Z_{24}^{1/2} \theta_{124} \theta_{234} \\
&\left. - \delta^{ad}\delta^{bc} Z_{14}^{1/2} Z_{23}^{1/2} \theta_{123} \theta_{234} \right]
\end{aligned} \tag{47}$$

The first term has no counterpart in the bosonic theory as it is a consequence of the extra term in (46). The other three terms are the supersymmetric equivalent of (33). When we fuse $K^{sc}(Z_3)$ and $K^{sd}(Z_4)$ in (47), the extra term only contributes to the 2-point function $\langle |J^{sa}(Z_1)J^{sb}(Z_2)| \rangle$, and so we find

$$\langle |J^{si}(Z_1)J^{sj}(Z_2)N^{sk}(Z_3)| \rangle = \frac{-i\epsilon^{ijk}\theta_{123}}{Z_{12}^{1/2}Z_{23}^{1/2}Z_{31}^{1/2}} \quad (48)$$

From these correlation functions, we can extract the following OPEs:

$$\begin{aligned} K^{si}(Z_1)K^{sj}(Z_2) &= \frac{\delta^{ij}}{Z_{12}^2} - \frac{i\epsilon^{ijk}\theta_{12}}{Z_{12}^2}N^{sk}(Z_2) + \frac{i\epsilon^{ijk}}{Z_{12}}DN^{sk} + \dots \\ J^{si}(Z_1)K^{sj}(Z_2) &= \frac{i\epsilon^{ijk}\theta_{12}}{Z_{12}}K^{sb} + \dots \\ J^{si}(Z_1)N^{sj}(Z_2) &= \frac{\delta^{ij}}{Z_{12}} + \frac{\theta_{12}}{Z_{12}}i\epsilon^{ijk}N^{sk}(Z_2) + \dots \end{aligned} \quad (49)$$

Comparing (49) and (46), it can be seen that in the case of the SWZW model we can consistently identify $N^{sa}(Z)$ with $J^{sa}(Z)$. However, we cannot know if this is correct without knowing the OPE of N^{sa} with itself, for which we would need to know six-point and eight-point correlation functions in the SWZW model. To rewrite these OPEs in terms of bosonic and fermionic component fields, we note that J^{sa} and N^{sa} have superconformal dimension 1/2, and K^{sa} has dimension 1. In other words, the bosonic components all have conformal dimension 1, but the fermionic component of K^s has dimension 3/2 while the fermionic components of J^s and N^s have dimension 1/2. We can therefore decompose J^s , K^s and N^s as:

$$\begin{aligned} J^{sa}(Z) &= \theta J^a(z) + j^a(z) \\ K^{sa}(Z) &= K^a(z) + \theta k^a(z) \\ N^{sa}(Z) &= \theta N^a(z) + n^a(z) \end{aligned} \quad (50)$$

The OPE for the component fields is then:

$$J^a(z_1)J^b(z_2) = \frac{\delta^{ab}}{z_{12}^2} + \frac{i\epsilon^{abc}}{z_{12}}J^c(z_2) + \dots \quad (51)$$

$$K^a(z_1)K^b(z_2) = \frac{\delta^{ab}}{z_{12}^2} + \frac{i\epsilon^{abc}}{z_{12}}N^c(z_2) + \dots \quad (52)$$

$$J^a(z_1)K^b(z_2) = \frac{i\epsilon^{abc}}{z_{12}}K^c(z_2) + \dots \quad (53)$$

$$J^a(z_1)N^b(z_2) = \frac{\delta^{ab}}{z_{12}^2} + \frac{i\epsilon^{abc}}{z_{12}}N^c(z_2) + \dots \quad (54)$$

$$j^a(z_1)j^b(z_2) = \frac{\delta^{ab}}{z_{12}} + \dots \quad (55)$$

$$k^a(z_1)k^b(z_2) = \frac{2\delta^{ab}}{z_{12}^3} + \dots \quad (56)$$

$$j^a(z_1)k^b(z_2) = \frac{-i\epsilon^{abc}}{z_{12}}J^c(z_2) + \dots \quad (57)$$

$$j^a(z_1)n^b(z_2) = \frac{\delta^{ab}}{z_{12}} + \dots \quad (58)$$

$$j^a(z_1)J^b(z_2) = \frac{-i\epsilon^{abc}}{z_{12}}j^c(z_2) + \dots \quad (59)$$

$$j^a(z_1)K^b(z_2) = \dots \quad (60)$$

$$j^a(z_1)N^b(z_2) = \frac{-i\epsilon^{abc}}{z_{12}}n^c(z_2) + \dots \quad (61)$$

$$J^a(z_1)k^b(z_2) = \frac{i\epsilon^{abc}}{z_{12}}k^c(z_2) + \dots \quad (62)$$

$$J^a(z_1)n^b(z_2) = \frac{i\epsilon^{abc}}{z_{12}}n^c(z_2) + \dots \quad (63)$$

$$K^a(z_1)k^b(z_2) = \frac{i\epsilon^{abc}}{z_{12}^2}n^c(z_2) + \frac{i\epsilon^{abc}}{z_{12}}\partial n^c(z_2) + \dots \quad (64)$$

If we identify $J^{sa} = N^{sa}$, the bosonic components reduce to an $SO(4)_2$ current algebra:

$$\begin{aligned} J^a(z_1)J^b(z_2) &= \frac{\delta^{ab}}{z_{12}^2} + \frac{i\epsilon^{abc}}{z_{12}}J^c(z_2) + \dots \\ K^a(z_1)K^b(z_2) &= \frac{\delta^{ab}}{z_{12}^2} + \frac{i\epsilon^{abc}}{z_{12}}J^c(z_2) + \dots \\ J^a(z_1)K^b(z_2) &= \frac{i\epsilon^{abc}}{z_{12}}K^c(z_2) + \dots \end{aligned} \quad (65)$$

However, the fermionic components are not the same as in an $SO(4)$ SWZW theory.

5 Conclusion

In this paper we have found new interesting features in $SU(N)$ level $k = 0$ WZNW model and $SU(N)$ level $k = N$ supersymmetric WZNW models using as toy models the $N = 2$ cases. Besides logarithmic operators, there are new additional conserved currents generating new hidden symmetries. In the theories we have discussed, there are no operators with negative conformal dimensions, and so, in contrast to the majority of theories with logarithmic operators, these

theories are not non-unitary. The further investigation of these symmetries is of great interest.

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